The Basics of Set Theory

L545

Spring 2013

Based on Partee, ter Meulen, & Wall (1993), *Mathematical Methods in Linguistics*

Why set theory?

Set theory sets the foundation for much of mathematics

- For us: provides precise ways to define/describe (types of) models for linguistic analysis
- The concepts here are fundamental for any further work in CS or CL

You've seen some of this before, but we'll systematize it

Sets

A **set** is a collection of objects

- $A = \{a, b\}$ designates the set A
- $a \in A$ means a is a member of A
- $c \notin A$ means c is not a member of A
- $|A| = 2$ denotes the **cardinality**, or size, of set A

Other ways to specify the same set:

- $A = \{a, a, b, a, b, b\}$ ... in other words, sets do not have repeats
- $A = \{x | x$ is a letter of the alphabet before $c\}$

NB: $\emptyset$ designates the empty set, i.e., set with no members

Subsets

If every member of a set A is a member of a set B, then A is a **subset** of B, denoted $A \subseteq B$

- B could also be equal to A by this definition, i.e., a set can be a subset of itself
- To state that B contains more members ($A \neq B$), we say that A is a proper subset of B, written $A \subset B$
- If A contains a member that B does not, then A is not a subset of B, written $A \not\subset B$

Some examples (Partee et al, p. 10):

- $\{a, b, c\} \subseteq \{s, b, a, e, g, i, c\}$
- $\{a, b, j\} \not\subseteq \{s, b, a, e, g, i, c\}$
- $\emptyset \subseteq \{a\}$
- $\{a, \{a\}\} \subseteq \{a, b, \{a\}\}$
- $\{a\} \not\subseteq \{\{a\}\}$ (but $\{a\} \in \{\{a\}\}$)

Power sets

The **power set** of a set A is the set of all subsets of A and is denoted $\wp(A)$ or 2^A

- If $A = \{a, b\}$, then $\wp(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- $|\wp(A)| = 2^{|A|}$

Power sets are often used in definitions

Union and intersection

The operations to be most familiar with are **union** and **intersection**

- Union: $A \cup B =_{def} \{x | x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B =_{def} \{x | x \in A \text{ and } x \in B\}$

Assume $K = \{a, b\}$, $L = \{c, d\}$, and $M = \{b, d\}$:

$K \cup L = \{a, b, c, d\}$ $\qquad$ $K \cap L = \emptyset$

$K \cup M = \{a, b, d\}$ $\qquad$ $K \cap M = \{b\}$

$(K \cup L) \cup M = K \cup (L \cup M) = \{a, b, c, d\}$

$(K \cap L) \cap M = K \cap (L \cap M) = \emptyset$

Difference and complement

Set **difference** “subtracts” out members in one set but not another

- $A - B =_{def} \{x | x \in A \text{ and } x \notin B\}$

Assume $K = \{a, b\}$, $L = \{c, d\}$, and $M = \{b, d\}$:

- $K - M = \{a\}$
- $L - K = \{c, d\} = L$

A set **complement** (A' or $\bar{A}$) is everything not in set, defined relative to the universe (U) of objects

- $A' =_{def} \{x | x \notin A\} = U - A$

Set-theoretic equalities (1)

1. *Idempotent Laws*
   (a) $X \cup X = X$ (b) $X \cap X = X$
2. *Commutative Laws*
   (a) $X \cup Y = Y \cup X$ (b) $X \cap Y = Y \cap X$
3. *Associative Laws*
   (a) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ (b) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$
4. *Distributive Laws*
   (a) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
   (b) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

Set-theoretic equalities (2)

5. *Identity Laws*
   (a) $X \cup \emptyset = X$ (c) $X \cap \emptyset = \emptyset$
   (b) $X \cup U = U$ (d) $X \cap U = X$
6. *Complement Laws*
   (a) $X \cup X' = U$ (c) $X \cap X' = \emptyset$
   (b) $(X')' = X$ (d) $X - Y = X \cap Y'$
7. *DeMorgan's Laws*
   (a) $(X \cup Y)' = X' \cap Y'$ (b) $(X \cap Y)' = X' \cup Y'$
8. *Consistency Principle*
   (a) $X \subseteq Y$ iff $X \cup Y = Y$ (b) $X \subseteq Y$ iff $X \cap Y = X$

Ordered pairs

Sets have no order to their elements, but we often want to establish an order; this is how we define **ordered pairs**:

- $< a, b >= \{\{a\}, \{a, b\}\}$
- It follows that $< a, b > \neq < b, a >$
- Definition can be extended to *n*-tuples

The **Cartesian product** of sets A and B is defined as all ordered pairs derived from those sets:

- $A \times B =_{def} \{< x, y > | x \in A \text{ and } y \in B\}$
- If $K = \{a, b, c\}$ and $L = \{1, 2\}$, then $K \times L = \{< a, 1 >, < a, 2 >, < b, 1 >, < b, 2 >, < c, 1 >, < c, 2 >\}$
- Note, though, that the ordered pairs within $K \times L$ are not ordered with respect to each other

Relations

A **relation** is simply a set of ordered pairs, and can be defined (for two sets A and B) as a subset of $A \times B$

- A relation $R \subseteq K \times L$ might be defined as: $\{< a, 1 >, < b, 1 >, < c, 1 >\}$
- Intuitively, we can define relations such as *mother-of* as consisting of <mother, child> pairs

Terminology:

- The **domain** is the set of all first terms and the **range** the set of all second terms
- We say that R is a relation *from A to B*

Functions

A **function** is a special type of relation, where:

1. Each element in the domain is paired with just one element in the range.
2. The domain of R is equal to A

Assume $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Functions:

- $P = \{< a, 1 >, < b, 2 >, < c, 3 >\}$
- $Q = \{< a, 3 >, < b, 4 >, < c, 1 >\}$
- $R = \{< a, 3 >, < b, 2 >, < c, 2 >\}$

Not functions:

- $S = \{< a, 1 >, < b, 2 >\}$
- $T = \{< a, 2 >, < b, 3 >, < a, 3 >, < c, 1 >\}$
- $V = \{< a, 2 >, < a, 3 >, < b, 4 >\}$

Properties: reflexivity

Given a set A and a relation R in A (i.e., $R \subseteq A \times A$):

- R is **reflexive** iff all the ordered pairs $< x, x >$ are in R, for every x in A
  - If $A = \{1, 2, 3\}$, then $R_1 = \{< 1, 1 >, < 2, 2 >, < 3, 3 >, < 3, 1 >\}$ is reflexive
  - $R_2 = \{< 1, 1 >, < 2, 2 >\}$ is nonreflexive
- R is **irreflexive** iff it contains no ordered pair $< x, x >$ with identical first & second members

Properties: symmetry

Given a set A and a relation R in A:

- R is **symmetric** iff for every ordered pair $< x, y >$ in R, the pair $< y, x >$ is also in R
  - e.g., $\{< 2, 3 >, < 3, 2 >, < 2, 2 >\}$ is symmetric
  - e.g., $\{< 2, 3 >, < 2, 2 >\}$ is nonsymmetric
- R is **asymmetric** iff it is never the case that for any $< x, y >$ in R, $< y, x >$ is in R
  - e.g., $\{< 2, 3 >, < 1, 2 >\}$
- R is **anti-symmetric** if whenever both $< x, y >$ and $< y, x >$ are in R, then $x = y$
  - e.g., $\{< 2, 3 >, < 1, 1 >\}$

The Basics of Set Theory
Sets
Operations
Equalities
Ordered pairs
Relations
Properties
Orderings

Properties: transitivity

Given a set A and a relation R in A:

- R is **transitive** iff for all ordered pairs $< x, y >$ and $< y, z >$ in R, $< x, z >$ is also in R
  - e.g., $\{< 1, 2 >, < 2, 3 >, < 1, 3 >, < 2, 2 >\}$ is transitive
  - e.g., $\{< 2, 3 >, < 3, 2 >, < 2, 2 >\}$ is nontransitive
- R is **intransitive** if for no pairs $< x, y >$ and $< y, z >$ in R, $< x, z >$ is in R
  - e.g., $\{< 3, 1 >, < 1, 2 >, < 2, 3 >\}$

The Basics of Set Theory
Sets
Operations
Equalities
Ordered pairs
Relations
Properties
Orderings

Properties: connectedness

Given a set A and a relation R in A:

- R is **connected** iff for every two *distinct* elements x and y in A, $< x, y > \in R$ or $< y, x > \in R$ (or both)
  - If $A = \{1, 2, 3\}$, $\{< 1, 2 >, < 3, 1 >, < 3, 2 >\}$ is connected
  - $\{< 1, 3 >, < 3, 1 >, < 2, 2 >, < 3, 2 >\}$ is nonconnected

The Basics of Set Theory
Sets
Operations
Equalities
Ordered pairs
Relations
Properties
Orderings

Orderings

An **order** is a binary relation which is *transitive* and either

(i) *reflexive* and *antisymmetric* (**weak order**) or
(ii) *irreflexive* and *asymmetric* (**strong order**)

- Essentially, cycles are disallowed
- antisymmetry & asymmetry differ in whether reflexive relations are allowed

If $A = \{a, b, c, d\}$:

- Strong order example: $S = \{< a, b >, < a, c >, < a, d >, < b, c >\}$
- Weak order example: $R = \{< a, b >, < a, c >, < a, d >, < b, c >, < a, a >, < b, b >, < c, c >, < d, d >\}$
- If the order is connected, it is a **total order**; otherwise, a **partial order**

The Basics of Set Theory
Sets
Operations
Equalities
Ordered pairs
Relations
Properties
Orderings