

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

The Basics of Set Theory

L445 / L545

Spring 2017

Based on Partee, ter Meulen, & Wall (1993),
Mathematical Methods in Linguistics

Why set theory?

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

Set theory sets the foundation for much of mathematics

- ▶ For us: provides precise ways to define/describe (types of) models for linguistic analysis
- ▶ The concepts here are fundamental for any further work in CS or CL

You've seen some of this before, but we'll systematize it

A **set** is a collection of objects

- ▶ $A = \{a, b\}$ designates the set A
- ▶ $a \in A$ means a is a member of A
- ▶ $c \notin A$ means c is not a member of A
- ▶ $|A| = 2$ denotes the **cardinality**, or size, of set A

Other ways to specify the same set:

- ▶ $A = \{a, a, b, a, b, b\}$... in other words, sets do not have repeats
- ▶ $A = \{x \mid x \text{ is a letter of the alphabet before } c\}$

NB: \emptyset designates the empty set, i.e., set with no members

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

If every member of a set A is a member of a set B , then A is a **subset** of B , denoted $A \subseteq B$

- ▶ B could also be equal to A by this definition, i.e., a set can be a subset of itself
- ▶ To state that B contains more members ($A \neq B$), we say that A is a **proper subset** of B , written $A \subset B$
- ▶ If A contains a member that B does not, then A is not a subset of B , written $A \not\subseteq B$

Some examples (Partee et al, p. 10):

- ▶ $\{a, b, c\} \subseteq \{s, b, a, e, g, i, c\}$
- ▶ $\{a, b, j\} \not\subseteq \{s, b, a, e, g, i, c\}$
- ▶ $\emptyset \subseteq \{a\}$
- ▶ $\{a, \{a\}\} \subseteq \{a, b, \{a\}\}$
- ▶ $\{a\} \not\subseteq \{\{a\}\}$ (but $\{a\} \in \{\{a\}\}$)

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

The **power set** of a set A is the set of all subsets of A and is denoted $\wp(A)$ or 2^A

- ▶ If $A = \{a, b\}$, then $\wp(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- ▶ $|\wp(A)| = 2^{|A|}$

Power sets are often used in definitions

The operations to be most familiar with are **union** and **intersection**

- ▶ Union: $A \cup B =_{\text{def}} \{x \mid x \in A \text{ or } x \in B\}$
- ▶ Intersection: $A \cap B =_{\text{def}} \{x \mid x \in A \text{ and } x \in B\}$

Assume $K = \{a, b\}$, $L = \{c, d\}$, and $M = \{b, d\}$:

$$K \cup L = \{a, b, c, d\} \qquad K \cap L = \emptyset$$

$$K \cup M = \{a, b, d\} \qquad K \cap M = \{b\}$$

$$(K \cup L) \cup M = K \cup (L \cup M) = \{a, b, c, d\}$$

$$(K \cap L) \cap M = K \cap (L \cap M) = \emptyset$$

Set **difference** “subtracts” out members in one set but not another

$$\blacktriangleright A - B =_{\text{def}} \{x \mid x \in A \text{ and } x \notin B\}$$

Assume $K = \{a, b\}$, $L = \{c, d\}$, and $M = \{b, d\}$:

$$\blacktriangleright K - M = \{a\}$$

$$\blacktriangleright L - K = \{c, d\} = L$$

A set **complement** (A' or \bar{A}) is everything not in set, defined relative to the universe (U) of objects

$$\blacktriangleright A' =_{\text{def}} \{x \mid x \notin A\} = U - A$$

Set-theoretic equalities (1)

1. Idempotent Laws

$$(a) X \cup X = X \quad (b) X \cap X = X$$

2. Commutative Laws

$$(a) X \cup Y = Y \cup X \quad (b) X \cap Y = Y \cap X$$

3. Associative Laws

$$(a) (X \cup Y) \cup Z = X \cup (Y \cup Z) \quad (b) (X \cap Y) \cap Z = X \cap (Y \cap Z)$$

4. Distributive Laws

$$(a) X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

$$(b) X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$

Set-theoretic equalities (2)

5. Identity Laws

$$(a) X \cup \emptyset = X \quad (c) X \cap \emptyset = \emptyset$$

$$(b) X \cup U = U \quad (d) X \cap U = X$$

6. Complement Laws

$$(a) X \cup X' = U \quad (c) X \cap X' = \emptyset$$

$$(b) (X')' = X \quad (d) X - Y = X \cap Y'$$

7. DeMorgan's Laws

$$(a) (X \cup Y)' = X' \cap Y' \quad (b) (X \cap Y)' = X' \cup Y'$$

8. Consistency Principle

$$(a) X \subseteq Y \text{ iff } X \cup Y = Y \quad (b) X \subseteq Y \text{ iff } X \cap Y = X$$

Ordered pairs

Sets have no order to their elements, but we often want to establish an order; this is how we define **ordered pairs**:

- ▶ $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$
- ▶ It follows that $\langle a, b \rangle \neq \langle b, a \rangle$
- ▶ Definition can be extended to n -tuples

The **Cartesian product** of sets A and B is defined as all ordered pairs derived from those sets:

- ▶ $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}$
- ▶ If $K = \{a, b, c\}$ and $L = \{1, 2\}$, then $K \times L = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$
- ▶ Note, though, that the ordered pairs within $K \times L$ are not ordered with respect to each other

Sets

Operations
Equalities

Ordered pairs

Relations
Properties
Orderings

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

A **relation** is simply a set of ordered pairs, and can be defined (for two sets A and B) as a subset of $A \times B$

- ▶ A relation $R \subseteq K \times L$ might be defined as:
 $\{ \langle a, 1 \rangle, \langle b, 1 \rangle, \langle c, 1 \rangle \}$
- ▶ Intuitively, we can define relations such as *mother-of* as consisting of $\langle \text{mother}, \text{child} \rangle$ pairs

Terminology:

- ▶ The **domain** is the set of all first terms and the **range** the set of all second terms
- ▶ We say that R is a relation *from* A *to* B

A **function** is a special type of relation, where:

1. Each element in the domain is paired with just one element in the range.
2. The domain of R is equal to A

Assume $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Functions:

- ▶ $P = \{ \langle a, 1 \rangle, \langle b, 2 \rangle, \langle c, 3 \rangle \}$
- ▶ $Q = \{ \langle a, 3 \rangle, \langle b, 4 \rangle, \langle c, 1 \rangle \}$
- ▶ $R = \{ \langle a, 3 \rangle, \langle b, 2 \rangle, \langle c, 2 \rangle \}$

Not functions:

- ▶ $S = \{ \langle a, 1 \rangle, \langle b, 2 \rangle \}$
- ▶ $T = \{ \langle a, 2 \rangle, \langle b, 3 \rangle, \langle a, 3 \rangle, \langle c, 1 \rangle \}$
- ▶ $V = \{ \langle a, 2 \rangle, \langle a, 3 \rangle, \langle b, 4 \rangle \}$

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

Properties: reflexivity

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

Given a set A and a relation R in A (i.e., $R \subseteq A \times A$):

- ▶ R is **reflexive** iff all the ordered pairs $\langle x, x \rangle$ are in R , for every x in A
 - ▶ If $A = \{1, 2, 3\}$, then
 - $R_1 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3 \rangle, \langle 3, 1 \rangle\}$ is reflexive
 - $R_2 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$ is nonreflexive
- ▶ R is **irreflexive** iff it contains no ordered pair $\langle x, x \rangle$ with identical first & second members

Properties: symmetry

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

Given a set A and a relation R in A :

- ▶ R is **symmetric** iff for every ordered pair $\langle x, y \rangle$ in R , the pair $\langle y, x \rangle$ is also in R
 - ▶ e.g., $\{\langle 2, 3 \rangle, \langle 3, 2 \rangle, \langle 2, 2 \rangle\}$ is symmetric
 - ▶ e.g., $\{\langle 2, 3 \rangle, \langle 2, 2 \rangle\}$ is nonsymmetric
- ▶ R is **asymmetric** iff it is never the case that for any $\langle x, y \rangle$ in R , $\langle y, x \rangle$ is in R
 - ▶ e.g., $\{\langle 2, 3 \rangle, \langle 1, 2 \rangle\}$
- ▶ R is **anti-symmetric** if whenever both $\langle x, y \rangle$ and $\langle y, x \rangle$ are in R , then $x = y$
 - ▶ e.g., $\{\langle 2, 3 \rangle, \langle 1, 1 \rangle\}$

Properties: transitivity

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

Given a set A and a relation R in A :

- ▶ R is **transitive** iff for all ordered pairs $\langle x, y \rangle$ and $\langle y, z \rangle$ in R , $\langle x, z \rangle$ is also in R
 - ▶ e.g., $\{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle\}$ is transitive
 - ▶ e.g., $\{\langle 2, 3 \rangle, \langle 3, 2 \rangle, \langle 2, 2 \rangle\}$ is nontransitive
- ▶ R is **intransitive** if for no pairs $\langle x, y \rangle$ and $\langle y, z \rangle$ in R , $\langle x, z \rangle$ is in R
 - ▶ e.g., $\{\langle 3, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle\}$

Properties: connectedness

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings

Given a set A and a relation R in A :

- ▶ R is **connected** iff for every two *distinct* elements x and y in A , $\langle x, y \rangle \in R$ or $\langle y, x \rangle \in R$ (or both)
 - ▶ If $A = \{1, 2, 3\}$:
 - ▶ $\{\langle 1, 2 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$ is connected
 - ▶ $\{\langle 1, 3 \rangle, \langle 3, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 2 \rangle\}$ is nonconnected

An **order** is a binary relation which is *transitive* and either

- (i) *reflexive* and *antisymmetric* (**weak order**) or
 - (ii) *irreflexive* and *asymmetric* (**strong order**)
- ▶ Essentially, cycles are disallowed
 - ▶ antisymmetry & asymmetry differ in whether reflexive relations are allowed

If $A = \{a, b, c, d\}$:

- ▶ Strong order example:
 $S = \{ \langle a, b \rangle, \langle a, c \rangle, \langle a, d \rangle, \langle b, c \rangle \}$
- ▶ Weak order example: $R = \{ \langle a, b \rangle, \langle a, c \rangle, \langle a, d \rangle, \langle b, c \rangle, \langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle \}$
- ▶ If the order is connected, it is a **total order**;
otherwise, a **partial order**

Sets

Operations

Equalities

Ordered pairs

Relations

Properties

Orderings