1 All the rules

Here are all the natural deduction rules for intuitionistic propositional logic (and intuitionistic predicate calculus without quantifiers).

\[
\begin{align*}
A & \rightarrow B & A \land B \rightarrow C & \rightarrow I \\
\frac{A \land B \rightarrow C}{A \land B \rightarrow C} & \rightarrow E_L \\
\frac{A \land B \rightarrow C}{A \land B \rightarrow C} & \rightarrow E_R \\
\frac{A \lor B \rightarrow C}{A \lor B \rightarrow C} & \lor I_L \\
\frac{A \lor B \rightarrow C}{A \lor B \rightarrow C} & \lor I_R \\
\frac{\bot \rightarrow B}{A \rightarrow B} & \Rightarrow I \\
\frac{\bot \rightarrow B}{A \rightarrow B} & \Rightarrow E \\
\frac{\bot \rightarrow B}{A \rightarrow B} & \Rightarrow E \\
\frac{\bot \rightarrow B}{A \rightarrow B} & \Rightarrow E \\
\frac{\bot \rightarrow B}{A \rightarrow B} & \Rightarrow E
\end{align*}
\]
2 Rule variations

2.1 Getting rid of negation

If we use the definition $\neg \phi = \phi \Rightarrow \bot$, then we can see that the $\bot I$ rule is just the same thing as $\Rightarrow I$ (where $B = \bot$) and the $\bot E$ rule is just the same thing as $\Rightarrow E$ (where $B = \bot$). Thus, in order to reduce the number of rules we have to worry about, we could get rid of the $\bot I$ and $\bot E$ rules by using the given definition for negation, rather than considering negation to be a primitive logic.

2.2 The ATP rules

Some of the rules given in Pfenning’s Automated Theorem Proving book are slightly different from the ones presented here. In particular, these are the versions he has differently.

The motivation for these rule variations is that they help to make derivations shorter. We can think of the $\lor E^{u,v}$ rule as just an abbreviation for the following derivation which uses our original rules.

In the $\neg l^{u,C}$ rule we require that $C$ is parametric. That is, the proof of $A \vdash C$ must work for all $C$ and cannot assume anything about $C$. In other words, instead of knowing what $C$ is and then giving the proof $A \vdash C$ as in normal proofs, we have to give a generic proof of $A \vdash C$ such that someone can plug in any possible $C$ afterwards.
Another way of thinking about parametricity is that in System F we can give the definitions
\( \bot = (\forall X. X) \) and \( \top = (\exists X. X) \). The idea being that we can’t possibly have a proof of \((\forall X. X)\) since that means we have a proof of every possible proposition! Of course, due to the principle of explosion (aka \( \bot E \)), if we have a proof of \( \bot \) then we do in fact have a proof of anything we please. On the other side \((\exists X. X)\) is just saying that there’s some proposition which is provable. We should hope that at least one possible proposition is true, so we can take it as axiomatic that \((\exists X. X)\) is true (aka the \(\top I\) rule). And by convention we’ll call the provable proposition “\(\top\)”. But since we know nothing about \(\top\) other than that it’s provable, we can’t come up with any sensible elimination rule for it (since elimination is a way of taking apart propositions that we know the structure of).

Even if parametricity doesn’t make sense to you, we can still make sense of these rules as just ways of abbreviating or rearranging our original rules. In particular, the ATP derivations on the left below are equal to the derivations on the right which use our original rules. For negation introduction, because the ATP version must prove a parametric \(C\), we know it has to derive \(\bot\) somewhere along the way and then use \(\bot E\) to get the \(C\). Thus, we can just get rid of the \(\bot E\) step and use our original \(\neg I\). (In particular, the \(\bot E\) step has been moved over to \(\neg E\).)

\[
\begin{align*}
\frac{A \text{ true}^u}{\bot} \quad \vdots \quad \frac{\bot \text{ true} \quad \bot E}{C \text{ true} \quad \neg A} & \quad \Rightarrow \quad \frac{A \text{ true}}{\bot \text{ true} \quad \neg I_u C} \quad \frac{\bot \text{ true}}{\neg A} \\
\frac{A \text{ true}}{\neg A \text{ true} \quad \neg E} & \quad \Leftrightarrow \quad \frac{\bot \text{ true} \quad \bot E}{\bot E} \\
\end{align*}
\]

### 3 Classical rules

For classical logic, we need to add one of these additional rules: Proof by Contradiction (\(\bot_C\)), Double Negation elimination (\(\neg \neg E_C\)), or the Law of Excluded Middle (\(\text{LEM}_C\), or \(\text{XM}_C\)). It doesn’t matter which is chosen because the others can be derived from it. For example, if we have double negation elimination then we can derive proof by contradiction, by using the \((\neg A \text{ true} \vdash \bot \text{ true})\) deduction as the premise in \(\neg I\) (thus yielding an intermediate conclusion \(\neg \neg A \text{ true}\)) and then using \(\neg \neg E_C\) to get \(A \text{ true}\).

\[
\begin{align*}
\frac{\neg A \text{ true}^u}{\bot} \quad \vdots \quad \frac{\bot \text{ true} \quad \bot C}{A \text{ true} \quad \neg I_C} & \quad \frac{\neg A \text{ true} \quad \neg E_C}{A \text{ true} \quad \neg C} \quad \frac{A \text{ true} \quad \neg A \text{ true}}{A \lor \neg A \text{ true} \quad \text{LEM}_C} \\
\end{align*}
\]
4 Examples

4.1 Associativity of Conjunction

\[
\begin{array}{c}
A \land (B \land C) \text{ true} \\
A \text{ true} \quad \land_E \\
B \land C \text{ true} \quad \land_E \\
B \text{ true} \quad \land_I \\
(A \land B) \text{ true} \quad \land_I \\
\end{array}
\]

4.2 Associativity of Disjunction

For want of wider pages, let \( \mathcal{D} \) be the following derivation:

\[
\begin{array}{c}
B \lor C \text{ true} \\
B \text{ true} \quad \lor_I \\
A \lor B \text{ true} \quad \lor_I \\
A \Rightarrow (A \lor B) \lor C \text{ true} \quad \Rightarrow I^u \\
C \Rightarrow (A \lor B) \lor C \text{ true} \quad \Rightarrow I^v \\
(A \lor B) \lor C \text{ true} \quad \lor_E \\
\end{array}
\]

And now we can use \( \mathcal{D} \) in the larger proof.

\[
\begin{array}{c}
A \lor (B \lor C) \text{ true} \\
A \text{ true} \quad \lor_I \\
A \Rightarrow (A \lor B) \lor C \text{ true} \quad \Rightarrow I^u \\
B \lor C \text{ true} \quad \lor_I \\
B \Rightarrow (A \lor B) \lor C \text{ true} \quad \Rightarrow I^v \\
(A \lor B) \lor C \text{ true} \quad \lor_E \\
\end{array}
\]

4.3 De Morgan’s law 1: \( \neg(A \lor B) \Rightarrow \neg A \land \neg B \)

Let this proof be called \( \mathcal{E} \), for reuse in a later proof.

\[
\begin{array}{c}
\neg(A \lor B) \text{ true} \\
A \text{ true} \quad \lor_I \\
A \Rightarrow (A \lor B) \text{ true} \quad \Rightarrow I^u \\
\downarrow \text{ true} \quad \neg I^u \\
\neg A \text{ true} \quad \neg E \\
\end{array}
\]

\[
\begin{array}{c}
\neg(A \lor B) \text{ true} \\
A \lor B \text{ true} \quad \lor_I \\
\downarrow \text{ true} \quad \neg I^u \\
\neg A \text{ true} \quad \neg E \\
\end{array}
\]

\[
\begin{array}{c}
\neg(A \lor B) \text{ true} \\
B \text{ true} \quad \lor_I \\
\downarrow \text{ true} \quad \neg I^u \\
\neg B \text{ true} \quad \neg E \\
\end{array}
\]

\[
\begin{array}{c}
\neg(A \lor B) \text{ true} \\
\neg A \land \neg B \text{ true} \quad \land_E \\
\end{array}
\]
### 4.4 De Morgan’s law 2: $\neg A \land \neg B \Rightarrow \neg (A \lor B)$

Using the original rules we can almost fit this on the page.

\[
\begin{array}{c}
\text{\textit{A true}}^v \\
\text{\neg A true}^u \\
\hline
\neg A \land \neg B \text{ true}^w \\
\hline
\text{\textit{E}}^E \\
\hline
\text{\neg (A \lor B) true}^\neg I
\end{array}
\]

And here’s the same proof, using the ATP rule for disjunction elimination.

\[
\begin{array}{c}
\text{\textit{A true}}^v \\
\text{\neg A true}^u \\
\hline
\neg (A \land B) \text{ true}^w \\
\hline
\text{\neg I}\neg C
\end{array}
\]

### 4.5 De Morgan’s law 3: $\neg (A \land B) \Rightarrow \neg A \lor \neg B$

N.B., this one requires classical axioms. The other three of De Morgan’s laws are intuitionistic, but this one is classical for the same reason that $\neg (\forall x. \phi) \Rightarrow (\exists x. \neg \phi)$ is classical. And for brevity we’ll reuse the proof $\&$ of De Morgan’s law for converting negation over disjunction into conjunction over negation.

\[
\begin{array}{c}
\text{\neg A \land B \text{ true}}^u \\
\text{\neg A true}^v \\
\hline
\neg A \lor B \text{ true}^w \\
\hline
\neg I\neg C
\end{array}
\]

### 4.6 De Morgan’s law 4: $\neg A \lor \neg B \Rightarrow \neg (A \land B)$

This time we’ll just show the version using the ATP rule for disjunction elimination.
\[ \neg A \lor \neg B \text{ true} \]

\[ \underline{\neg A \text{ true}} \quad \underline{A \text{ true}} \quad \underline{\neg B \text{ true}} \]

\[ \underline{A \land B \text{ true}} \quad \underline{\neg B \text{ true}} \]

\[ \neg A \lor \neg B \text{ true} \]

\[ \neg (A \land B) \text{ true} \]