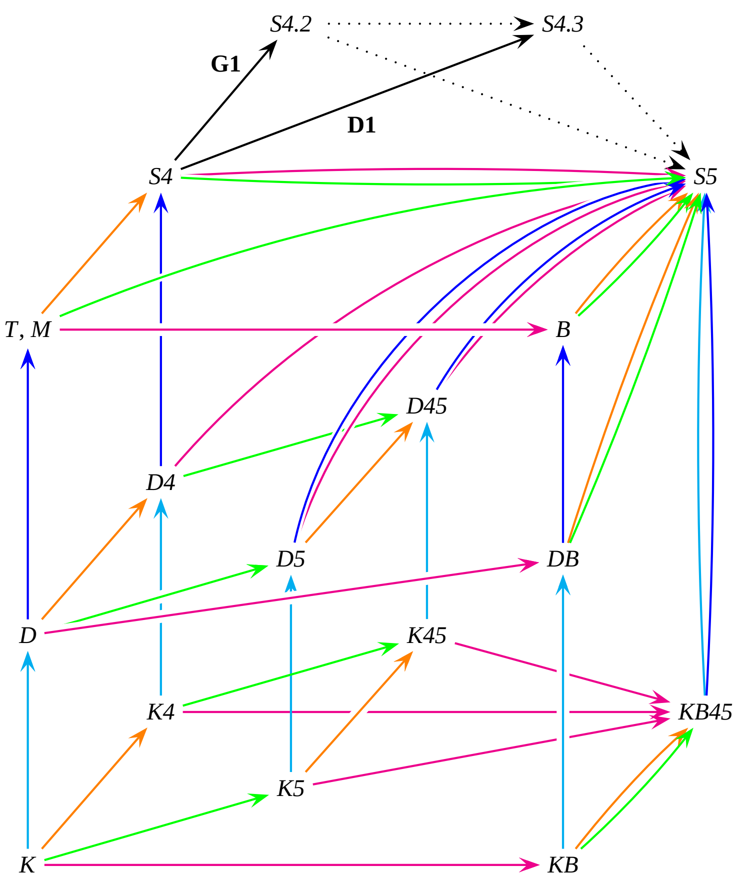
N	if $\vdash \phi$ then $\vdash \Box\phi$		Necessitation Rule	
K	$\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$	$\Box(\phi \to \psi) \to \Diamond\phi \to \Diamond\psi$	Normality Axioms	
C	$\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$	$\Diamond(\phi \vee \psi) \to \Diamond\phi \vee \Diamond\psi$	(by Normality and Necessitation)	
M	$\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$	$\Diamond\phi \vee \Diamond\psi \to \Diamond(\phi \vee \psi)$	(by Normality and Necessitation)	
?	$\Box\phi \vee \Box\psi \to \Box(\phi \vee \psi)$	$\Diamond(\phi \wedge \psi) \to \Diamond\phi \wedge \Diamond\psi$	(by Normality and Necessitation)	
Scott–Lemmon	$\Diamond^h\Box^i\phi \to \Box^j\Diamond^k\phi$	$\Diamond^j\Box^k\phi \to \Box^h\Diamond^i\phi$	(h,i,j,k)-diamond property	$\forall w,x,y.\ wR^hx \wedge wR^jy \Rightarrow \exists z.\ xR^iz \wedge yR^kz$
T, **M**	$\Box\phi \to \phi$	$\phi \to \Diamond\phi$	reflexive	$\forall x.\ xRx$
B	$\phi \to \Box\Diamond\phi$	$\Diamond\Box\phi \to \phi$	symmetry	$\forall x,y.\ xRy \Rightarrow yRx$
B′	$\Box(\psi \vee \Box\phi) \to \phi \vee \Box\psi$	$\phi \wedge \Diamond\psi \to \Diamond(\psi \wedge \Diamond\phi)$	symmetry	$\forall x,y.\ xRy \Rightarrow yRx$
4	$\Box\phi \to \Box\Box\phi$	$\Diamond\Diamond\phi \to \Diamond\phi$	transitive	$\forall x,y,z.\ xRy \wedge yRz \Rightarrow xRz$
4$_c$	$\Box\Box\phi \to \Box\phi$	$\Diamond\phi \to \Diamond\Diamond\phi$	dense	$\forall x,z.\ xRz \Rightarrow \exists y.\ xRy \wedge yRz$
5	$\Diamond\phi \to \Box\Diamond\phi$	$\Diamond\Box\phi \to \Box\phi$	(right) Euclidean	$\forall x,y,z.\ xRy \wedge xRz \Rightarrow yRz$
D$_c$	$\Diamond\phi \to \Box\phi$	$\Diamond\phi \to \Box\phi$	functional	$\forall x,y,z.\ xRy \wedge xRz \Rightarrow y = z$
D	$\Box\phi \to \Diamond\phi$	*self-dual*	serial	$\forall x.\ \exists y.\ xRy$
D′	$\Box\neg\phi \to \neg\Box\phi$	*self-dual*	serial	$\forall x.\ \exists y.\ xRy$
G1, **C**	$\Diamond\Box\phi \to \Box\Diamond\phi$	*self-dual*	"convergent"; i.e., confluent	$\forall w,x,y.\ wRx \wedge wRy \Rightarrow \exists z.\ xRz \wedge yRz$
MS, **M**	$\Box\Diamond\phi \to \Diamond\Box\phi$	*self-dual*	?	(has no first-order frame condition)
□**T**, □**M**, **H**	$\Box(\Box\phi \to \phi)$		(right) shift-reflexive	$\forall x,y.\ xRy \Rightarrow yRy$
D1	$\Box(\Box\phi \to \psi) \vee \Box(\Box\psi \to \phi)$		"connected"; i.e., pre-Euclidean	$\forall x,y,z.\ xRy \wedge xRz \Rightarrow yRz \vee zRy$
GL, **L**, **W**	$\Box(\Box\phi \to \phi) \to \Box\phi$		transitive and well-founded	(has no first-order frame condition)



$$\Box(\pi) = \mathbf{K} \mathbin{\$} \mathbf{N}(\pi)$$

$$\mathbf{T} = \mathbf{B}^{op} \circ \mathbf{D} \circ \mathbf{4}$$
$$\mathbf{B} = \mathbf{5} \circ \mathbf{T}^{op}$$
$$\mathbf{4} = \Box(\mathbf{5}^{op}) \circ \mathbf{B}$$
$$\mathbf{5} = \Box(\mathbf{4}^{op}) \circ \mathbf{B}$$
$$\mathbf{D} = \mathbf{T}^{op} \circ \mathbf{T}$$

$$\mathbf{4} = \ldots\mathbf{GL}\ldots$$
$$\mathbf{4}_c = \mathbf{K} \mathbin{\$} \mathbf{H}$$
$$\mathbf{D1} = \ldots\mathbf{5}\ldots$$

Correspondence between N/K and $\Box$I/$\Box$E

Let $\_ \,\$\, \_$ denote the modus ponens rule (associating to the left); let $\llbracket \_ \rrbracket$ denote $\Box$I; and, let $!\_$ denote $\Box$E. For any context $C : \phi \vdash \psi$ and any proof $p : \Gamma \vdash \Box\phi$ we have the following functorality theorem:

$$\llbracket \_[!\_] \rrbracket \;:\; (\phi \vdash \psi) \;\Rightarrow\; (\Gamma \vdash \Box\phi) \;\Rightarrow\; (\Gamma \vdash \Box\psi)$$
$$\llbracket C[!p] \rrbracket = \mathbf{K} \,\$\, \mathbf{N}(\lambda x.\, C[x]) \,\$\, p$$

If you have multiple proofs $p_i : \Gamma \vdash \Box\phi_i$, then just repeat the process as necessary; e.g.,

$$\llbracket C[!p,\, !q] \rrbracket = \mathbf{K} \,\$\, \mathbf{K} \,\$\, \mathbf{N}(\lambda x\, y.\, C[x,y]) \,\$\, p \,\$\, q$$

Thus, the $\Box$I rule corresponds to the **N** rule, and the $\Box$E rule corresponds to the **K** axiom.

Example proofs

K entails C

1	$\Box\phi \wedge \Box\psi$	
2	$\Box\phi$	$\wedge$E: 1
3	$\Box\psi$	$\wedge$E: 1
4	$\Box(\phi \to \psi \to \phi \wedge \psi)$	**N**
5	$\Box\phi \to \Box(\psi \to \phi \wedge \psi)$	$\to$E: **K**, 4
6	$\Box(\psi \to \phi \wedge \psi)$	$\to$E: 5, 2
7	$\Box\psi \to \Box(\phi \wedge \psi)$	$\to$E: **K**, 6
8	$\Box(\phi \wedge \psi)$	$\to$E: 7, 3
9	$\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$	$\to$I: 1–8

1	$\Box\phi \wedge \Box\psi$	
2	$\Box\phi$	$\wedge$E: 1
3	$\Box\psi$	$\wedge$E: 1
4	$\star$ ϕ	$\Box$E: 2
5	ψ	$\Box$E: 3
6	$\phi \wedge \psi$	$\wedge$I: 4, 5
7	$\Box(\phi \wedge \psi)$	$\Box$I: 4–6
8	$\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$	$\to$I: 1–7

K entails M

1	$\Box(\phi \wedge \psi)$	
2	$\star$ $\phi \wedge \psi$	$\Box$E: 1
3	ϕ	$\wedge$E: 2
4	$\Box\phi$	$\Box$I: 2–3
5	$\star$ $\phi \wedge \psi$	$\Box$E: 1
6	ψ	$\wedge$E: 5
7	$\Box\psi$	$\Box$I: 5–6
8	$\Box\phi \wedge \Box\psi$	$\wedge$I: 4, 7
9	$\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$	$\to$I: 1–8

1	$\Box(\phi \wedge \psi)$	
2	$\Box(\phi \wedge \psi \to \phi)$	**N**
3	$\Box(\phi \wedge \psi) \to \Box\phi$	$\to$E: **K**, 2
4	$\Box\phi$	$\to$E: 3, 1
5	$\Box(\phi \wedge \psi \to \psi)$	**N**
6	$\Box(\phi \wedge \psi) \to \Box\psi$	$\to$E: **K**, 5
7	$\Box\psi$	$\to$E: 6, 1
8	$\Box\phi \wedge \Box\psi$	$\wedge$I: 4, 7
9	$\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$	$\to$I: 1–8

Just a helpful proof to have around

1	$\Box\phi \land \Diamond\psi$	
2	$\Box\neg(\phi \land \psi)$	
3	$\Box\phi$	$\land$E: 1
4	$\star$ ϕ	$\Box$E: 3
5	$\neg(\phi \land \psi)$	$\Box$E: 2
6	$\neg\phi \lor \neg\psi$	De Morgan: 5
7	$\neg\psi$	Disjunctive Syllogism: 4, 6
8	$\Box\neg\psi$	$\Box$I: 4–7
9	$\Diamond\psi$	$\land$E: 1
10	$\neg\Box\neg\psi$	R: 9
11	$\bot$	$\neg$E: 10, 8
12	$\neg\Box\neg(\phi \land \psi)$	$\neg$I: 2–11
13	$\Diamond(\phi \land \psi)$	R: 12
14	$\Box\phi \land \Diamond\psi \to \Diamond(\phi \land \psi)$	$\to$I: 1–13

GL entails 4

1	$\Box\phi$	
2	$\star$ ϕ	$\Box$E: 1
3	$\Box(\phi \land \Box\phi)$	
4	$\Box\phi \land \Box\Box\phi$	$\to$E: **M**, 3
5	$\Box\phi$	$\land$E: 4
6	$\phi \land \Box\phi$	$\land$I: 2, 5
7	$\Box(\phi \land \Box\phi) \to \phi \land \Box\phi$	$\to$I: 3–6
8	$\Box(\Box(\phi \land \Box\phi) \to \phi \land \Box\phi)$	$\Box$I: 2–7
9	$\Box(\phi \land \Box\phi)$	$\to$E: **GL**, 8
10	$\Box\phi \land \Box\Box\phi$	$\to$E: **M**, 9
11	$\Box\Box\phi$	$\land$E: 10
12	$\Box\phi \to \Box\Box\phi$	$\to$I: 1–11