1. Let \( R = \text{it is raining}, L = \text{Logic is fun}, \) and \( S = \text{Superman exists}. \)

   a. Translate the following propositions into the most natural equivalent statements in English.

      i. \((R \lor S) \implies L\)

         If it is raining or if Superman exists, then logic is fun.

      ii. \((L \land \neg R) \lor (\neg L \land R)\)

         Either logic is fun or it is raining, but not both.

      iii. \((R \lor L) \iff S\)

         It is raining or logic is fun if and only if Superman exists.

      iv. \(S \implies (\neg L \lor R)\)

         If Superman exists, then it is either raining or logic is not fun or both.

      v. \((R \land S \land \neg L) \lor (L \land \neg R \land \neg S)\)

         Either it is raining and Superman exists or logic is fun, but not both.

      vi. \((R \lor \neg S) \land (R \lor S)\)

         It is raining.

   b. Translate the following statements into propositional logic.

      i. Superman does not exist and it is raining, but logic is fun.
\[\neg S \land \neg R \land L\]

ii. Logic is fun only if it is raining.

\[L \Rightarrow R\]

iii. In order for Superman to exist, it is necessary and sufficient for it to be raining or for logic to be fun, or both.

\[S \leftrightarrow R \lor L\]

iv. Exactly one of the following statements is true: Superman exists, it is not raining, and logic is fun.

\[(S \land \neg L \land \neg R) \lor (\neg S \land L \land \neg R) \lor (\neg S \land \neg L \land R)\]

v. Whenever it is raining, Superman exists or logic is fun, but not both.

\[R \Rightarrow S \oplus L\]

vi. For Superman to exist it is sufficient but not necessary for logic to be fun.

\[L \Rightarrow S\]

vii. Whether or not Superman exists, logic is fun.

\[L\]

viii. Logic is not fun if Superman exists or if it is raining.

\[S \lor R \Rightarrow \neg L\]

ix. If Superman exists, then it must be raining in order for logic to be fun.

\[S \Rightarrow (L \Rightarrow R)\]

x. Superman exists if it is raining and logic is fun.

\[R \land L \Rightarrow S\]

2. Construct truth tables for the following propositions.

a. \[p \Rightarrow (\neg q \land p)\]

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p \Rightarrow (\neg q \land p)</th>
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<tr>
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b. \((p \land q) \iff (\neg r \lor q)\)

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c. \((p \lor r \lor q) \Rightarrow (q \iff \neg p)\)

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<th>p</th>
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d. \((p \land r) \lor (\neg p \Rightarrow \neg r)\)

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e. \((p \land r \lor \neg q) \iff (p \land q)\)

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3. Determine whether each of the following propositions is a tautology, contradiction, or contingency. If it is a contingency, provide two interpretations to demonstrate this is the case.

a. \((p \land q) \iff (\neg r \lor q)\)

  **contingency:**
  - \(p = True, q = True, r = True\) makes it True
  - \(p = False, q = True, r = True\) makes it False

b. \((p \land r) \lor (p \rightarrow \neg r)\)

  **tautology**

c. \(((p \land \neg q) \rightarrow (r \lor q)) \iff (p \land \neg q \land r)\)

  **contingency:**
  - \(p = True, q = False, r = True\) makes it True
  - \(p = True, q = True, r = True\) makes it False

d. \(\neg(a \lor b \land \neg c) \lor (a \lor b \land \neg c)\)

  **contingency:**
  - \(a = False, b = True, c = True\) makes it False
  - All other interpretations make it True

e. \((q \rightarrow p) \land (\neg s \rightarrow \neg q) \land \neg(p \rightarrow s) \land q\)

  **contingency:**
  - \(p = True, q = False, s = False\) makes it True
  - All other interpretations make it False

f. \(\neg(p \lor r) \land (\neg p \Rightarrow r \land s)\)

  **contradiction**

4. For each of the following, determine whether the assertion is a logical consequence of the given premises. If the assertion is not a logical consequence, explain why not.

a. Premises:
   - \(p \lor \neg q\)
   - \(\neg r \Rightarrow (p \lor q)\)
   - \(r \lor p\)
Assertion:
q ∨ r

Not a logical consequence: When p = True, q = False, r = False the premises are True but the assertion is False.

b. Premises:
   (∼q ∧ ∼r) ⇒ p
   ∼r ⇔ (p ∨ q)
   (r ∧ p) ⇒ ∼q
   Assertion:
   q

Not a logical consequence: When p = True, q = False, r = False the premises are True but the assertion is False.

c. Premises:
   p ∧ ∼q
   q ∨ r
   Assertion:
   ∼p ∨ q ∨ r

   Logical consequence

d. Premises:
   p ∨ ∼q
   p ⇒ q
   Assertion:
   p

   Not a logical consequence: When p = False, q = False the premises are True but the assertion is False.

e. Premises:
   p ∨ ∼q ∧ r
   r ⇒ (q ∧ ∼p)
   q
   Assertion:
   ∼r ∧ q

   Logical consequence
5. Let Loves(x, y) mean “x loves y”, Student(x) mean “x is a student”, Friend(x, y) mean “x is a friend of y”, Bicycle(x) mean “x is a bicycle”, Owns(x, y) mean “x owns y”, and Sister(x, y) mean “x is a sister of y”.

a. Translate the following propositions into the most natural equivalent statements in English.

i. \( \forall x \exists y \text{Student}(x) \implies \text{Sister}(y, x) \)

All students have at least one sister.

ii. \( \text{Student}(\text{Bill}) \land \exists x \text{Bicycle}(x) \land \text{Owns}(\text{Bill}, x) \land \forall y (y \neq \text{Bill}) \iff \neg \text{Friend}(y, \text{Bill}) \)

Bill is a student who owns a bicycle and whose only friend is himself.

iii. \( \exists x \exists y \forall z (z \neq x) \implies \text{Friend}(x, z) \land \text{Bicycle}(y) \land \text{Owns}(x, y) \)

There is at least one person who is friends with all others and who owns a bicycle.

iv. \( \forall x \exists y \text{Student}(x) \implies \text{Friend}(x, y) \land \forall z \text{Friend}(x, z) \implies (y = z) \)

Every student has exactly one friend.

v. \( \forall x \forall y \exists z \text{Student}(x) \land \text{Bicycle}(z) \land \text{Owns}(x, z) \implies \text{Friend}(x, y) \)

Every student who owns a bicycle is friends with everyone, including himself/herself.

vi. \( \forall x \text{Student}(x) \implies \exists y \text{Bicycle}(y) \land \text{Owns}(x, y) \land \text{Loves}(x, y) \)

Every student owns a bicycle that he or she loves.

vii. \( \forall y \text{Bicycle}(y) \implies \exists x \text{Student}(x) \land \text{Owns}(x, y) \)

Every bicycle is owned by at least one student.

viii. \( \forall x \exists y \text{Student}(x) \land \text{Bicycle}(y) \land \text{Owns}(x, y) \implies \exists w \text{Sister}(x, w) \)

Every student who owns a bicycle has at least one sister.

b. Translate the following statements into predicate logic.
i. Every student has a sister or a bicycle (or both).

$$\forall x \text{Student}(x) \Rightarrow \exists y \text{SisterOf}(y, x) \lor \exists z \text{Bicycle}(z) \land \text{Owns}(x, z)$$

ii. If a student is a friend of all of his or her sisters, then that student owns a bicycle.

$$\forall x (\text{Student}(x) \land \forall y \text{SisterOf}(x, y) \Rightarrow \text{Friend}(x, y)) \Rightarrow \exists z \text{Bicycle}(z) \land \text{Owns}(x, z)$$

iii. There is at least one student who is loved by everyone but is a friend of only himself/herself.

$$\exists x \text{Student}(x) \land \forall y \text{ Loves}(y, x) \land \forall z \text{Friend}(x, z) \iff (x = z)$$

iv. Every student has at least one friend and one bicycle.

$$\forall x \exists y \exists z \text{Student}(x) \Rightarrow \text{Friend}(x, y) \lor (\text{Bicycle}(z) \land \text{Owns}(x, z))$$

v. Bill loves Sally and is a friend of everyone who owns a bicycle.

$$\text{Loves}(\text{Bill}, \text{Sally}) \land \forall x \exists y \text{Bicycle}(y) \land \text{Owns}(x, y) \Rightarrow \text{Friend}(\text{Bill}, x)$$

vi. Peter has only one friend but at least two bicycles.

$$\exists w \text{Friend}(\text{Peter}, w) \land \forall x \text{Friend}(\text{Peter}, x) \Rightarrow (w = x)$$

vii. Every student has at least one friend if and only if they own a bicycle.

viii. If something is a student it cannot be a bicycle, and vice versa.

$$\forall x \text{Student}(x) \Rightarrow \neg \text{Bicycle}(x)$$

ix. Any student that loves his or her bicycle has at least two friends.

$$\forall w \exists x \text{Student}(w) \land \text{Bicycle}(x) \land \text{Owns}(w, x) \Rightarrow \exists y \exists z \text{Friend}(w, y) \land \text{Friend}(w, z) \land (y \neq z)$$

x. All bicycle-loving sisters are students.
∀ x ∃ y ∃ z Sister(x, y) ^ Bicycle(z) ^ Owns(x, z) ⇒ Student(x)

6. For the following propositions, the universe of discourse is the set of all real numbers. Determine whether each of the following propositions is true or false, and explain your answer.

a. ∃ n (n^2 = 2)
   True, n = √2

b. ∀ x ∃ y (x = y^2)
   False, when x is a negative number there is no value for y that makes the equality true (because the square root of a negative number is not a real number.)

c. ∀ x ∃ y (xy = 1)
   False, when x = 0 there is no value for y which makes the equality true.

d. ∃ x (x^2 = -1)
   False, √-1 is not a real number.

e. ∀ x ∃ y (x + y = 1)
   True, y = 1 - x

f. ∀ x ∀ y ∃ z (z = (x + y)/2)
   True, the average of any two real numbers is also a real number.

7. For each of the following, prove that the conclusion follows logically from the premises.

a. Given:
   ¬f
   (g ^ h) ⇒ (f v e)
   (¬f v c) ⇒ h
   (¬f v b) ⇒ g
   Prove: e
1. \( \neg f \)  
2. \((g \land h) \Rightarrow (f \lor e)\)  
3. \((\neg f \lor c) \Rightarrow h\)  
4. \((\neg f \lor b) \Rightarrow g\)  
5. \(\neg f \lor c\) 1, addition  
6. \(h\) 3, 5 modus ponens  
7. \(\neg f \lor b\) 1, addition  
8. \(g\) 4, 7 modus ponens  
9. \(g \land h\) 6, 8 conjunction  
10. \(f \lor e\) 2, 9 modus ponens  
11. \(e\) 1, 10 disjunctive syllogism  

b. Given:  
\[ p \land (n \leftrightarrow s) \]  
\[ (p \lor r) \Rightarrow (q \land m) \]  
Prove:  
\[ p \land q \]  

1. \( p \land (n \leftrightarrow s) \)  
2. \((p \lor r) \Rightarrow (q \land m)\)  
3. \(p\) 1, simplification  
4. \(p \lor r\) 3, addition  
5. \(q \land m\) 2, 4 modus ponens  
6. \(q\) 5, simplification  
7. \(p \land q\) 3, 6 conjunction  

c. Given:  
\[ s \Rightarrow t \]  
\[ p \land q \]  
\[ (p \lor r) \Rightarrow s \]  
Prove:  
\[ t \]  

1. \( s \Rightarrow t \)  
2. \(p \land q\)  
3. \((p \lor r) \Rightarrow s\)  
4. \(p\) 2, simplification  
5. \(p \lor r\) 4, addition  
6. \(s\) 3, 5, modus ponens  
7. \(t\) 1, 6 modus ponens  

d. Given:  
\[ p \land q \]
\[(p \land r) \Rightarrow (t \land u)\]
\[r \land s\]
Prove:
\[t \lor v\]

1. \(p \land q\)
2. \((p \land r) \Rightarrow (t \land u)\)
3. \(r \land s\)

4. \(p\)  
   1, simplification
5. \(r\)  
   3, simplification
6. \(p \lor r\)  
   4, 5, conjunction
7. \(t \lor u\)  
   2, 6, modus ponens
8. \(t\)  
   7, simplification
9. \(t \lor v\)  
   8, addition

e. Given:
\[r \lor t\]
\[r \Rightarrow s\]
\[\neg t\]
\[(p \land q) \Leftrightarrow (r \land s)\]
Prove:
\[p\]

1. \(r \lor t\)
2. \(r \Rightarrow s\)
3. \(\neg t\)
4. \((p \land q) \Leftrightarrow (r \land s)\)
5. \(r\)  
   1, 3, disjunctive syllogism
6. \(s\)  
   2, 5, modus ponens
7. \(r \lor s\)  
   5, 6, conjunction
8. \(((p \land q) \Rightarrow (r \lor s)) \lor ((r \lor s) \Rightarrow (p \land q))\)
9. \((r \land s) \Rightarrow (p \land q)\)  
   8, simplification
10. \(p \lor q\)  
   7, 9, modus ponens
11. \(p\)  
   10, simplification

f. Given:
\[\neg q\]
\[p \lor q\]
\[p \Rightarrow (r \land s)\]
\[\neg u\]
\[r \Leftrightarrow (t \lor u)\]
Prove: \(t\)
1. \( \neg q \)
2. \( p \lor q \)
3. \( p \Rightarrow (r \land s) \)
4. \( \neg u \)
5. \( r \leftrightarrow (t \lor u) \)

6. \( p \quad 1, 2, \text{disjunctive syllogism} \)
7. \( r \land s \quad 3, 6, \text{modus ponens} \)
8. \( r \quad 7, \text{simplification} \)
9. \( (r \Rightarrow (t \lor u)) \land ((t \lor u) \Rightarrow r) \quad 5, \text{def. of biconditional} \)
10. \( r \Rightarrow (t \lor u) \quad 9, \text{simplification} \)
11. \( t \lor u \quad 8, 10, \text{modus ponens} \)
12. \( t \quad 4, 11, \text{disjunctive syllogism} \)

g. Given:
\((m \Rightarrow n) \Rightarrow (q \Rightarrow m)\)
\(m \lor q\)
\((m \lor q) \Rightarrow (m \Rightarrow n)\)
Prove:
\(n \lor m\)

1. \((m \Rightarrow n) \Rightarrow (q \Rightarrow m)\)
2. \(m \lor q\)
3. \((m \lor q) \Rightarrow (m \Rightarrow n)\)
4. \(m \Rightarrow n \quad 2, 3, \text{modus ponens} \)
5. \(q \Rightarrow m \quad 1, 4, \text{modus ponens} \)
6. \(n \lor m \quad 2, 4, 5 \text{ constructive dilemma} \)

h. Given:
\(w \Rightarrow x\)
\(\neg x\)
\(\neg w \Rightarrow (x \lor y)\)
Prove:
\(y\)

1. \(w \Rightarrow x\)
2. \(\neg x\)
3. \(\neg w \Rightarrow (x \lor y)\)
4. \(\neg w \quad 1, 2, \text{modus tollens} \)
5. \(x \lor y \quad 3, 4, \text{modus ponens} \)
6. \(y \quad 2, 5, \text{disjunctive syllogism} \)
i. Given:
\[ \neg(p \lor q) \Rightarrow \neg r \]
\[ \neg p \]
\[ r \]
Prove:
\[ q \]

1. \( \neg(p \lor q) \Rightarrow \neg r \)
2. \( \neg p \)
3. \( r \)
4. \( p \lor q \quad 1, 3, \text{modus tollens} \)
5. \( q \quad 2, 4, \text{disjunctive syllogism} \)

j. Given:
\( (\neg w \lor y) \Leftrightarrow (z \lor a) \)
\( w \Rightarrow \neg x \)
\( x \)
\( \neg a \)
Prove:
\( z \)

1. \( (\neg w \lor y) \Leftrightarrow (z \lor a) \)
2. \( w \Rightarrow \neg x \)
3. \( x \)
4. \( \neg a \)
5. \( \neg w \quad 2, 3, \text{modus tollens} \)
6. \( \neg w \lor y \quad 5, \text{addition} \)
7. \( (\neg w \lor y) \Rightarrow (z \lor a) \quad ^{\land} \)
\( (z \lor a) \Rightarrow (\neg w \lor y) \quad 1, \text{def. of biconditional} \)
8. \( (\neg w \lor y) \Rightarrow (z \lor a) \quad 7, \text{simplification} \)
9. \( z \lor a \quad 6, 8, \text{modus ponens} \)
10. \( z \quad 4, 9, \text{disjunctive syllogism} \)

k. Given:
\( w \Rightarrow x \)
\( w \Rightarrow \neg y \)
\( y \lor \neg x \)
Prove:
\( \neg w \)

1. \( w \Rightarrow x \)
2. \( w \Rightarrow \neg y \)
3. \( y \lor \neg x \)
4. \( w \) by assumption
5. \( x \) 1, 4, modus ponens
6. \( \neg y \) 2, 4, modus ponens
7. \( y \) 3, 5, disjunctive syllogism
8. \( y \land \neg y \) 6, 7, conjunction
Contradiction! Our assumption must be false and \( \neg w \) must be true.

l. Given:
\( (p \lor q) \Rightarrow r \)
\( \neg r \lor s \)
\( p \Rightarrow \neg s \)
Prove:
\( \neg p \)

1. \( (p \lor q) \Rightarrow r \)
2. \( \neg r \lor s \)
3. \( p \Rightarrow \neg s \)
4. \( p \) by assumption
5. \( \neg s \) 3, 4, modus ponens
6. \( \neg r \) 2, 5, disjunctive syllogism
7. \( p \lor q \) 4, addition
8. \( r \) 1, 7, modus ponens
9. \( r \land \neg r \) 6, 8, conjunction
Contradiction! Our assumption must be false and \( \neg p \) must be true.

m. Given:
\( \neg x \Rightarrow \neg z \)
y \( \Rightarrow z \)
Prove:
y \( \Rightarrow (x \lor w) \)

1. \( \neg x \Rightarrow \neg z \)
2. y \( \Rightarrow z \)
3. \( \neg (y \Rightarrow (x \lor w)) \) by assumption
4. \( \neg (\neg y \lor (x \lor w)) \) 3, def. of conditional
5. \( y \land \neg (x \lor w) \) 4, De Morgan’s
6. \( \neg (x \lor w) \) 5, simplification
7. \( \neg x \land \neg w \) 6, De Morgan’s
8. \( \neg x \) 7, simplification
9. y 5, simplification
10. \( z \) 2, 9, modus ponens
11. \( x \) 1, 10, modus tollens
12. \( x \land \neg x \) 8, 11, conjunction
Contradiction! Our assumption must be false and \( y \Rightarrow (x \lor w) \) must be true.

n. Given:
\[ p \iff (q \lor r) \]
\[ \neg q \]
Prove:
\[ p \Rightarrow (r \lor s) \]

1. \( p \iff (q \lor r) \)
2. \( \neg q \)
3. \( p \) by assumption
4. \( p \Rightarrow (q \lor r) \) \( \land \)
   \( (q \lor r) \Rightarrow p \) 1, def. of biconditional
5. \( p \Rightarrow (q \lor r) \) 4, simplification
6. \( (q \lor r) \) 3, 5, modus ponens
7. \( r \) 2, 6, disjunctive syllogism
8. \( r \lor s \) 7, addition

We have shown that when \( p \) is true, \( r \lor s \) must be true, so \( p \Rightarrow (r \lor s) \).

o. Given:
\[ q \iff s \]
\[ p \lor q \]
\[ p \Rightarrow r \]
\[ s \Rightarrow t \]
Prove:
\[ r \lor t \]

1. \( q \iff s \)
2. \( p \lor q \)
3. \( p \Rightarrow r \)
4. \( s \Rightarrow t \)
5. \( \neg (r \lor t) \) by assumption
6. \( \neg r \land \neg t \) 5, De Morgan's
7. \( \neg r \) 6, simplification
8. \( \neg p \) 3, 7, modus tollens
9. \( q \) 2, 8, disjunctive syllogism
10. \( s \) 1, 9, modus ponens
11. \( t \) 4, 10, modus ponens
12. \( \neg t \) 6, simplification
13. \( t \land \neg t \) 11, 12, conjunction

Contradiction! Our assumption must be false and \( (r \lor t) \) must be true.
8. Fred, Janice and Maggie are coworkers, about which you know the following. If Fred is not the highest paid of the three, then Janice is. If Janice is not the lowest paid, then Maggie is paid the most. Use this information to determine the relative salaries for the three coworkers (i.e., determine who is paid the most and who is paid the least).

Define the following propositions:

FH FM FL
JH JM JL
MH MM ML

The first letter represents the individual (F = Fred, J = Janice, M = Maggie) and the second letter represents where the salary of that individual ranks (H = highest, M = middle, L = low). So, e.g., FH is the proposition “Fred gets paid the highest amount.”

There are six possible person/salary combinations:

(1) FH ^ JM ^ ML
(2) FH ^ MM ^ JL
(3) JH ^ FM ^ ML
(4) JH ^ MM ^ FL
(5) MH ^ FM ^ JL
(6) MH ^ JM ^ FL

We are given the following information:

¬FH ⇒ JH
¬JL ⇒ MH

which can be rewritten as:

(1) FH ∨ JH
(2) JL ∨ MH

using the fact that p ⇒ q is equivalent to ¬ p ∨ q.

Next, we want to eliminate all of the possible solutions that violate either or these premises. Premise (1) rules out combinations (5) and (6), leaving:

(1) FH ^ JM ^ ML
(2) FH ^ MM ^ JL
and premise (2) rules out (1), (3), and (4), leaving:

(1) FH ^ JM ^ ML
(2) FH ^ MM ^ JL
(3) JH ^ FM ^ JL
(4) JH ^ MM ^ FL
(5) MH ^ FM ^ JL
(6) MH ^ JM ^ FL

So, combination (2) must be the solution: Fred is paid the most and Janice is paid the least.

9. A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook; the cook and the gardener cannot both be telling the truth; the gardener and the handyman are not both lying; and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, determine whether that person is telling the truth or lying.

Define the following propositions:

B = “The butler is telling the truth.”
C = “The cook is telling the truth.”
G = “The gardener is telling the truth.”
H = “The handyman is telling the truth.”

We are given the following information:

B ⇒ C
¬C v ¬G
G v H
H ⇒ ¬C

First, let’s assume that the butler is telling the truth and see what happens:

1. B ⇒ C
2. ¬C v ¬G
3. G v H
4. H ⇒ ¬C
5. B by assumption
Our assumption that the butler was telling the truth led to a contradiction, so we know that the butler must be lying (i.e., we know that \( \neg B \) must be true). So, we can now add this information to our premises. Next, let’s try assuming that the cook is telling the truth and see where that leads:

1. \( B \Rightarrow C \)
2. \( \neg C \lor \neg G \)
3. \( G \lor H \)
4. \( H \Rightarrow \neg C \)
5. \( \neg B \)
6. \( C \) by assumption
7. \( \neg H \) 4, 6, modus tollens
8. \( G \) 3, 7, disjunctive syllogism
9. \( \neg C \) 2, 8, disjunctive syllogism
10. \( C \land \neg C \) 6, 9, conjunction

Our assumption that the cook was telling the truth led to a contradiction, so we know that the cook must also be lying (i.e., we know that \( \neg C \) must be true). Again, we can add this information to our premises.

1. \( B \Rightarrow C \)
2. \( \neg C \lor \neg G \)
3. \( G \lor H \)
4. \( H \Rightarrow \neg C \)
5. \( \neg B \)
6. \( \neg C \)

Now, if we consider our premises again, we know from premise (3) that either the gardener or the handyman or both must be telling the truth. However, it is in fact not possible to determine whether the gardener or the handyman or neither are lying given only the information that we have. This could be demonstrated by constructing a truth table and showing that all of the premises are true whether or not \( H \) is true and whether or not \( G \) is true (as long as at least one of them is true), or you could just take my word for it. So, this was actually a trick question (sorry!) since we do not have enough information to determine whether or not the gardener or handyman are lying. So, the final result is:
The butcher and the cook are both lying, but we don't know whether or not the gardener or handyman are telling the truth.

10. The Game of Logic has the following two assumptions:
   I. “Logic is difficult or not many students like logic.”
   II. “If mathematics is easy, then logic is not difficult.”

By translating these assumptions into propositional statements, determine whether each of the following is a valid conclusion of these assumptions:
   a) Mathematics is not easy if many students like logic.
   b) Many students like logic if mathematics is not easy.
   c) Mathematics is not easy or logic is difficult.
   d) Logic is not difficult or mathematics is not easy.
   e) If not many students like logic, then either mathematics is not easy or logic is not difficult.

Define the following premises:

L = “Logic is difficult”
S = “Many students like logic”
M = “Math is easy”

Then our assumptions can be written as:

L v ¬S
M → ¬L

a) We want to determine if S → ¬M follows from the premises.

   1. L v ¬S
   2. M → ¬L
   3. ¬(S → ¬M) by assumption
   4. ¬(¬S v ¬M) 3, def. of conditional
   5. S ^ M 4, De Morgan’s
   6. S 5, simplification
   7. L 1, 6, disjunctive syllogism
   8. ¬M 2, 7, modus tollens
   9. M 5, simplification
   10. M ^ ¬M 8, 9, conjunction
Contradiction! Our assumption is false and S → ¬M must be true.

b) We want to determine whether ¬M → S follows from the premises. By constructing a truth table for both premises ((L v ¬S) v (M ⇒ ¬L)) and for the assertion (¬M ⇒ S) we find that there are interpretations where the premises are both true but the assertion is false (namely, the
interpretation \( L = \) True, \( M = \) False, \( S = \) True makes the premises True and the conclusion False.) So, \( \neg M \Rightarrow S \) is not a valid conclusion from the premises.

c) We want to determine whether \( \neg M \lor L \) follows from the premises.
Again, by constructing a truth table for both premises and the assertion (this time, \( \neg M \lor L \)) we find that there is an interpretation that makes both premises true and the assertion false (\( L = \) False, \( M = \) True, \( S = \) False). So, \( \neg M \lor L \) is not a valid conclusion.

d) We want to determine whether \( \neg L \lor \neg M \) follows from the premises.

1. \( L \lor \neg S \)
2. \( M \Rightarrow \neg L \)
3. \( \neg(\neg L \lor \neg M) \) by assumption
4. \( L \land M \) 3, De Morgan’s
5. \( L \) 4, simplification
6. \( M \) 4, simplification
7. \( \neg M \) 2, 5, modus tollens
8. \( M \land \neg M \) 6, 7, conjunction

Contradiction! So our assumption must be false and \( \neg L \lor \neg M \) must be true.

e) We want to determine whether \( \neg S \Rightarrow \neg M \lor \neg L \) follows from the premises.

1. \( L \lor \neg S \)
2. \( M \Rightarrow \neg L \)
3. \( \neg(\neg S \Rightarrow \neg M \lor \neg L) \) by assumption
4. \( \neg(S \lor (\neg M \lor \neg L)) \) 3, def. of conditional
5. \( \neg S \land \neg(\neg M \lor \neg L) \) 4, De Morgan’s
6. \( \neg S \land M \land L \) 5, De Morgan’s
7. \( M \) 6, simplification
8. \( L \) 6, simplification
9. \( \neg L \) 2, 7, modus ponens
10. \( L \land \neg L \) 8, 9, conjunction

Contradiction! So our assumption must be false and \( \neg S \Rightarrow \neg M \lor \neg L \) must be true.

11. Use mathematical induction to prove each of the following propositions about the positive integers.

a. \( (2(1) - 1) + (2(2) - 1) + ... + (2n - 1) = n^2 \)
Base case, $n = 1$:
$2(1) - 1 = 1$  
$1^2 = 1$

Inductive Hypothesis: Assume that the equation is true for some arbitrary $n$.

Inductive Step:
We want to prove $P(n+1)$:
$(2(1) - 1) + (2(2) - 1) + \ldots + (2(n) - 1) + (2(n+1)) = (n + 1)^2$

$(2(1) - 1) + (2(2) - 1) + \ldots + (2(n) - 1) + (2(n+1))$
$= n^2 + (2n +1)$  
by I.H.
$= (n + 1)^2$

b.  
$1^2 + 2^2 + \ldots + n^2 = (n(n+1)(2n+1))/6$

Base case, $n = 1$:
$1^2 = 1$  
$(1(1+1)(2(1)+1))/6 = (1(2)(3))/6 = 6/6 = 1$

Inductive Hypothesis: Assume that the equation is true for some arbitrary $n$.

Inductive Step:
We want to prove $P(n+1)$:
$1^2 + 2^2 + \ldots + n^2 + (n + 1)^2 = ((n+1)(n+2)(2n+3))/6$

$1^2 + 2^2 + \ldots + n^2 + (n + 1)^2$
$= (n(n+1)(2n+1))/6 + (n + 1)^2$  
by I.H.
$= (n(n+1)(2n+1) + 6(n+1)^2)/6$
$= ((n+1)(n(2n+1) + 6(n+1))/6$
$= ((n+1)(2n^2+7n+6))/6$
$= ((n+1)(n+2)(2n+3))/6$

c.  
$1^3 + 2^3 + \ldots + n^3 = (n^2(n +1)^2)/4$

Base case, $n = 1$:
$1^3 = 1$  
$(1^2(1 +1)^2)/4=((2)^2)/4=4/4=1$

Inductive Hypothesis: Assume that the equation is true for some arbitrary $n$.

Inductive Step:
We want to prove $P(n+1)$:
$1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = ((n+1)^2(n +2)^2)/4$

$1^3 + 2^3 + \ldots + n^3 + (n+1)^3$
\[ \frac{n^2(n+1)^2}{4} + (n+1)^3 \quad \text{by I.H.} \]
\[ = \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \]
\[ = \frac{((n+1)^2(n^2 + 4(n+1)))}{4} \]
\[ = \frac{(n+1)^2(n+2)^2}{4} \]

d. \[ 2^{1-1} + 2^{2-1} + \ldots + 2^{n-1} = 2^n - 1 \]

**Base case, n = 1:**
\[ 2^{1-1} = 2^0 = 1 \]
\[ 2^1 - 1 = 2 - 1 = 1 \]

**Inductive Hypothesis:** Assume that the equation is true for some arbitrary \( n \).

**Inductive Step:**
We want to prove \( P(n+1) \):
\[ 2^{1-1} + 2^{2-1} + \ldots + 2^{n-1} + 2^n = 2^{(n+1)} - 1 \]
\[ 2^{1-1} + 2^{2-1} + \ldots + 2^{n-1} + 2^n \]
= \[ 2^n - 1 + 2^n \quad \text{by I.H.} \]
= \[ 2(2^n) - 1 \]
= \[ 2^{(n+1)} - 1 \]
e. \[ n < 2^n \]

**Base case, n = 1:**
\[ 1 < 2^1 = 2 \]

**Inductive Hypothesis:** Assume that the equation is true for some arbitrary \( n \).

**Inductive Step:**
We want to prove \( P(n+1) \):
\[ (n+1) < 2^{(n+1)} \]
\[ (n+1) \]
\[ < 2^n + 1 \quad \text{by I.H.} \]
\[ = 2^n + 2^0 \]
\[ < 2^n + 2^n \]
\[ = 2^{(n+1)} \]